Série 3b Solutions

Exercise 3b.1 – Thermal effects

Consider the squared cross-section axial bar of Figure 3b.1. The thermal expansion coefficients of the material is $10 \cdot 10^{-6}\ K^{-1}$ and its Young modulus is 40 GPa. The initial temperature is room temperature (25°C). The tensile load is 480 N.

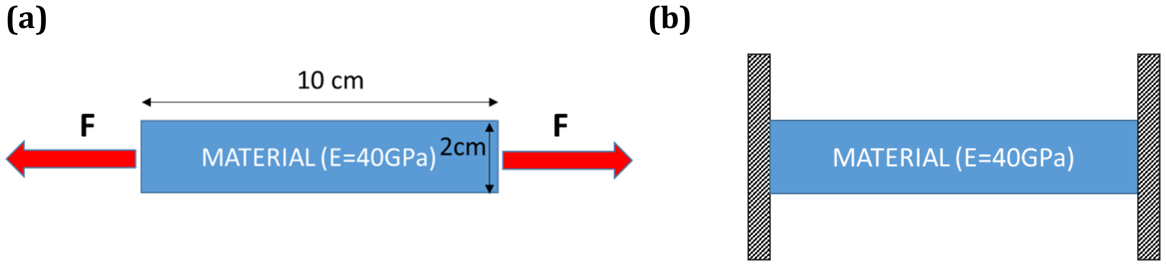


Figure 3b.1 | (a) Loaded bar, (b) Clamped bar.

- **We put the bar in a 500°C furnace (Figure 3b.1.a).**
  a) Determine the total longitudinal elongation
  b) Determine the strain energy density of the bar
- **Still in the furnace, we clamp the bar on its longitudinal ends (Figure 3b.1.b) and, from the furnace, put it in liquid Nitrogen (-200°C).**
  NB- <u>It is not anymore submitted to the external load F.</u>
  c) Determine the value of the induced stress due to temperature in the bar.

Solution 3b.1

What is given?

Thermal expansion coefficient $\alpha = 10 \cdot 10^{-6}\ K^{-1}$
Young Modulus E=40 GPa
Room temperature $T_0 = 25\ °C$
Furnace temperature $T_1 = 500\ °C$
Liquid nitrogen environment temperature: Room temperature $T_2 = -200\ °C$
Longitudinal cross-section of the material $A = 4\ cm^2$
Load $F = 480\ N$

Assumptions

The material is homogeneous and isotropic

What is asked?

a) Total longitudinal elongation

b) Stress energy density of the bar

c) Stress induced in the bar by cooling

Principles and formula

Hooke's law

$$\sigma = E\varepsilon \tag{1}$$

Where E is the Young modulus of the material that composes the bar, σ is the normal stress resulting from the load applied to the bar, and ε is the normal strain resulting from the load applied to the bar.

Strain definition

$$\varepsilon = \frac{\Delta L}{L_0} \tag{2}$$

ΔL is the normal deformation, and L_0 is the initial length of the bar.

Stress definition

$$\sigma = \frac{N}{A} \tag{3}$$

N is the internal force, and A is the cross section area.

Thermal strain

$$\varepsilon_{Th} = \alpha \Delta T \tag{4}$$

ε_{Th} is the thermal strain, α is the thermal expansion coefficient of the material, ΔT is the temperature variation between two states in thermal equilibria.

Strain energy density of the bar

$$U_0 = \frac{1}{2} E(\varepsilon_{tot} - \varepsilon_{Th})^2 \tag{5}$$

U_0 is the strain energy density of the bar, ε_{tot} is the total strain of the bar, and ε_{Th} is the thermal strain of the bar.

Calculations

a) In this section $\Delta T = T_1 - T_0$. We calculate the load deformation thanks to Hooke's law and stress and deformation definitions.

$$\sigma = E \cdot \varepsilon \rightarrow \frac{N}{A} = \frac{F}{A} = \frac{\Delta L}{L_0} \cdot E \rightarrow \Delta L = \frac{F \cdot L_0}{E \cdot A} \tag{6}$$

Using Eq. (6) and Eq. (4), we can calculate the total deformation, which is caused by both mechanical and thermal load. We apply the superposition principle and:

$$\Delta L_{tot} = \Delta L + \varepsilon_{Th} L_0 = \frac{F \cdot L_0}{E \cdot A} + \alpha L_0 \Delta T \tag{7}$$

b) We can now rewrite Eq. (5) in terms of strain:

$$\varepsilon_{tot} = \frac{\Delta L_{tot}}{L_0} = \frac{\Delta L}{L_0} + \varepsilon_{Th} = \frac{F}{E \cdot A} + \alpha \Delta T = \varepsilon_F + \varepsilon_{Th} \tag{8}$$

where ε_{Th} is the strain caused by the thermal expansion and ε_F is the strain caused by the load F. Therefore the strain energy is given by:

$$U_0 = \frac{1}{2} E (\varepsilon_{tot} - \varepsilon_{Th})^2 = \frac{1}{2} E \varepsilon_F^2 = \frac{1}{2} E \left(\frac{F}{E \cdot A}\right)^2 \tag{9}$$

c) In this section $\Delta T = T_2 - T_1$. Since the beam is clamped, we can see that the $\varepsilon_{tot} = 0$, thus:

$$\varepsilon_{tot} = \varepsilon_{mech} + \varepsilon_{Th} = \frac{\sigma}{E} + \varepsilon_{Th} = 0 \rightarrow \sigma = -E\varepsilon_{Th} = -E\alpha\Delta T \tag{10}$$

State your answer

a)

$$\Delta L_{tot} = \frac{F \cdot L_0}{E \cdot A} + \alpha(\Delta T) L_0 = \frac{480 \cdot 10 \cdot 10^{-2}}{40 \cdot 10^9 \cdot 4 \cdot 10^{-4}} + 10 \cdot 10^{-6} \cdot 475 \cdot 10 \cdot 10^{-2} = 478\ \mu m \tag{11}$$

b)

$$U_0 = \frac{1}{2} 40 \cdot 10^9 \cdot \left(\frac{480}{40 \cdot 10^9 \cdot 4 \cdot 10^{-4}}\right)^2 = 18\ \frac{J}{m^3} \tag{12}$$

c)

$$\sigma = -40 \cdot 10^9 \cdot 10 \cdot 10^{-6} \cdot (-700) = 280\ MPa \tag{13}$$

The *mechanical stress* that builds up in the bar when cooling down is 280 MPa ***tensile***.

Exercise 3b.2 – Plane Stress

A square plate of width b and thickness t is loaded by normal forces F_x and F_y, and by shear forces V, as shown in Figure 3b.2. These forces produce uniformly distributed stresses acting on the side faces of the plate. **Calculate the change in the volume $\Delta V = V_{final} - V_{initial}$ of the plate and strain energy U stored in the plate** if the dimensions are $b = 600\,mm$ and $t = 40\,mm$, the plate is made of magnesium with $E = 45\,GPa$ and $\nu = 0.35$, and the forces are $F_x = 480\,kN$, $F_y = 180\,kN$, and $V = 120\,kN$.

Formula for strain energy density in two dimensions.

$$u_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}\right) \tag{1}$$

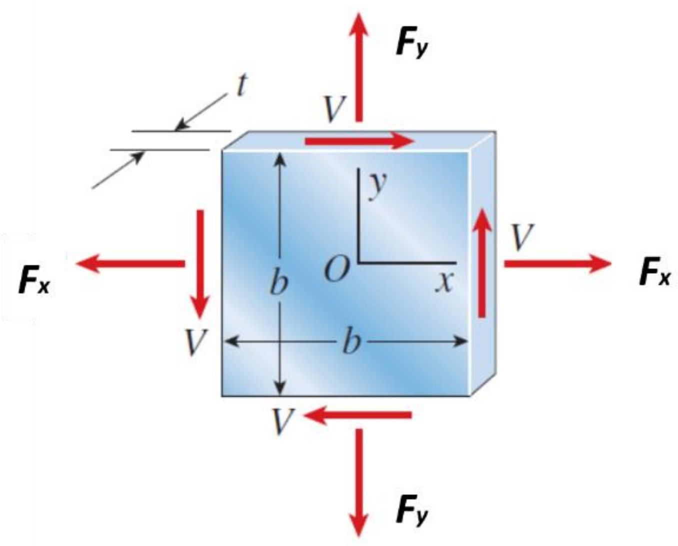


Figure 3b.2 | Loads on a cube

Solution 3b.2

Given:

A block with multiple normal and shear loads, see in figure 3b.2:

$b = 600\ mm$

$t = 40\ mm$

$E = 45\ GPa$

$\nu = 0.35$

$F_x = 480\ kN$

$F_y = 180\ kN$

$V = 120\ kN$

What is asked:

a) Change in volume, $V_{final} - V_{initial}$.

b) Strain energy stored in the block.

Solution:

a) The stresses in the material can be directly calculated from the loads, i.e.:

$$\sigma_x = \frac{N_x}{A} = \frac{F_x}{A} = \frac{F_x}{bt}; \sigma_y = \frac{F_y}{A} = \frac{F_y}{bt};\ \tau_{xy} = \frac{V}{A} = \frac{V}{bt} \tag{2}$$

As we know from theory, applying the strain equations from the compliance matrix to Eq. (2) we can derive the following equation:

$$\frac{V_{final} - V_{initial}}{V_{initial}} = \frac{(1-2\nu)}{E}\left(\sigma_x + \sigma_y + \sigma_z\right) \tag{3}$$

Inserting values for the initial volume $V_{initial} = b^2 \cdot t$, σ_x, and σ_y into Eq. (3) we obtain the following:

$$V_{final} - V_{initial} = \left(b^2 \cdot t\right) \cdot \left[\frac{(1-2\nu)}{E \cdot (t \cdot b)}\right]\left(F_x + F_y\right) = b\frac{(1-2\nu)}{E}\left(F_x + F_y\right) \tag{4}$$

$$V_{final} - V_{initial} = 600\ mm \cdot \frac{1 - 2 \cdot 0.35}{45\ GPa}(480\ kN + 180\ kN) = 4 \cdot 660 \cdot 10^{-9}\ m^3 \tag{5}$$

$$V_{final} - V_{initial} = 2.64 \cdot 10^{-6} m^3 \tag{6}$$

b) Using Eq. (1) and substituting strains with stresses with 3D Hooke's law, we are left with the following:

$$u_0 = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y\right) + \frac{\tau_{xy}^2}{2G} \tag{7}$$

$$G = \frac{E}{2(1+\nu)} \tag{8}$$

Inserting the stresses from Eq. (2) into (7), we get the following:

$$u_0 = \frac{1}{2E}\left(\left(\frac{F_x}{t \cdot b}\right)^2 + \left(\frac{F_y}{t \cdot b}\right)^2 - 2\nu\left(\frac{F_x}{t \cdot b}\right)\left(\frac{F_y}{t \cdot b}\right)\right) + \frac{1}{2G}\left(\frac{V}{t \cdot b}\right)^2 \tag{9}$$

$$u_0 = 4653\,Pa = 4653\,J/m^3 \tag{10}$$

Remember that u_0 is the strain energy *density*, so the final result is:

$$U = u_0 \cdot V_{initial} = 4653\frac{J}{m^3} \cdot 600^2\,mm^2 \cdot 40\,mm = 67\,J \tag{11}$$

Also keep in mind that the relative change in volume is very small so it is not going to make a difference whether we consider $V_{initial}$ or V_{final} in Eq. (11).

Exercise 3b.3 – Hybrid stiffness – 3D structure

A block of rubber (R on Figure 3b.3) is confined in a slot inside a steel block (S on Figure 3b.3). A uniform pressure p_0 applied on the top of the rubber block induces a deformation. The rubber's Young's modulus E and the rubber's Poisson's ratio ν are known.

a) Give an expression for the pressure along x axis on the block induced by p_0 and calculate its value

- **NB – We will neglect friction effects**

b) Give an expression for the dilatation e of the rubber and calculate its value

- **NB – The dilatation is also called the relative volume variation, i.e. $e = \frac{\Delta V}{V}$**

c) Find the strain-energy density u_0 of the rubber

Numerical values: $p_0 = 5.0\ MPa; E = 15.0\ GPa;\ \nu = 0.50$

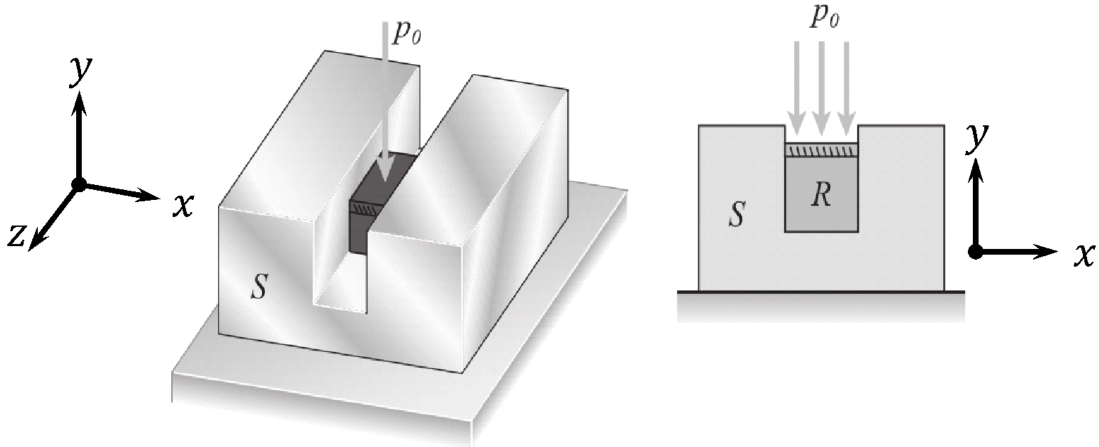


Figure 3b.3| Block of rubber in a steel block

Solution 3b.3

<u>What is given?</u>

Pressure $p_0 = 5.0\ MPa$

Young's modulus $E = 15.0\ GPa$

Poisson's ratio $\nu = 0.50$

<u>What is asked</u>

a) A formula for the lateral pressure on the block induced by p_0 and calculate its value
b) A formula for the dilatation e of the rubber and calculate its value
c) The strain-energy density u_0 of the rubber

<u>Equations required</u>

We will use the generalized Hooke's law, or compliance matrix:

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right);\ \gamma_{xy} = \frac{\tau_{xy}}{G}; etc\ ... \tag{1}$$

Where E is the Young's modulus of the material, ν is the Poisson's ratio, ε_x is the axial strain in the x-direction, and σ_x is the normal stress parallel to the x-axis, τ_{xy} and γ_{xy} are the shear stress and strain on the plane xy, and G is the shear modulus of the material. We then define the strain energy density in three dimensions:

$$u_0 = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) - \frac{\nu}{E}\left(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z\right) + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right) \tag{2}$$

We can also write the volume relative variation in relation with the stress components:

$$\frac{\Delta V}{V} = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right) \tag{3}$$

<u>Calculations</u>

a) The pressure is opposed to the internal stress of the material. We are looking for the pressure p along the x-direction.

$$p_x = -\sigma_x \tag{4}$$

We have been given the pressure in the y-direction.

$$p_0 = -\sigma_y \tag{5}$$

Then, no stress is induced in the z-direction, and being clamped, no strain can occur in the x direction:

$$\sigma_z = 0; \varepsilon_x = 0 \tag{6}$$

We apply the general Hooke's law.

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) \tag{7}$$

i.e. using (4), (5) and (6) in (7):

$$\varepsilon_x = 0 = -p_x - \nu\left(-p_0\right) \tag{8}$$

Thus,

$$p_x = \nu p_0 \tag{9}$$

b)

$$\frac{\Delta V}{V} = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{10}$$

i.e.

$$\frac{\Delta V}{V} = \frac{1-2\nu}{E}(-p_x - p_0) = -\frac{(1-2\nu)(1+\nu)p_0}{E} \tag{11}$$

c) Using the formula of the strain energy, we substitute the different components of stress in all the directions. There is no shear due to the wall constraint and only the y direction contributes:

$$u_0 = \frac{1}{2E}(1-\nu^2)p_0^2 \tag{12}$$

<u>State your answer</u>

a) The lateral pressure is:

$$p_x = \nu p_0 = 2.5\ MPa \tag{13}$$

b) The relative change in volume is:

$$\frac{\Delta V}{V} = -\frac{(1-2\nu)(1+\nu)p_0}{E} = 0 \tag{14}$$

There is no volume variation. The rubber keeps the same properties.

c) The strain energy density (in 3D) is:

$$u_0 = \frac{1}{2E}(1-\nu^2)p_0^2 = \frac{1}{2}\frac{1-0.5^2}{15\cdot 10^9}5^2\cdot 10^{12}\ Pa = 625\ Pa = 625\ \frac{N}{m^2} = 625\ \frac{J}{m^3} \tag{15}$$

Exercise 3b.4 – Bars and spring in series

A system *1)* is composed of two different bars while a similar system *2)* is instead formed by a spring and a bar, as shown in the Figure 3b.4 Both systems are loaded with forces in A, B and C, and the materials are considered isotropic.

a) Draw the Free Body Diagram of the two systems

Provided the numerical values for the systems:

1) $A = 3\ cm^2, A_* = 2\ cm^2, E = 25\ GPa, L = 10\ cm, F_1 = 30\ kN, F_2 = 45\ kN, F_3 = 75kN$

2) $A = 3\ cm^2, E = 25\ GPa, L = 10\ cm, F_1 = 30\ kN, F_2 = 45\ kN, F_3 = 75kN, k_s = 1 \cdot 10^8\ \frac{kg}{s^2}$

b) Calculate the deformation of the two different systems

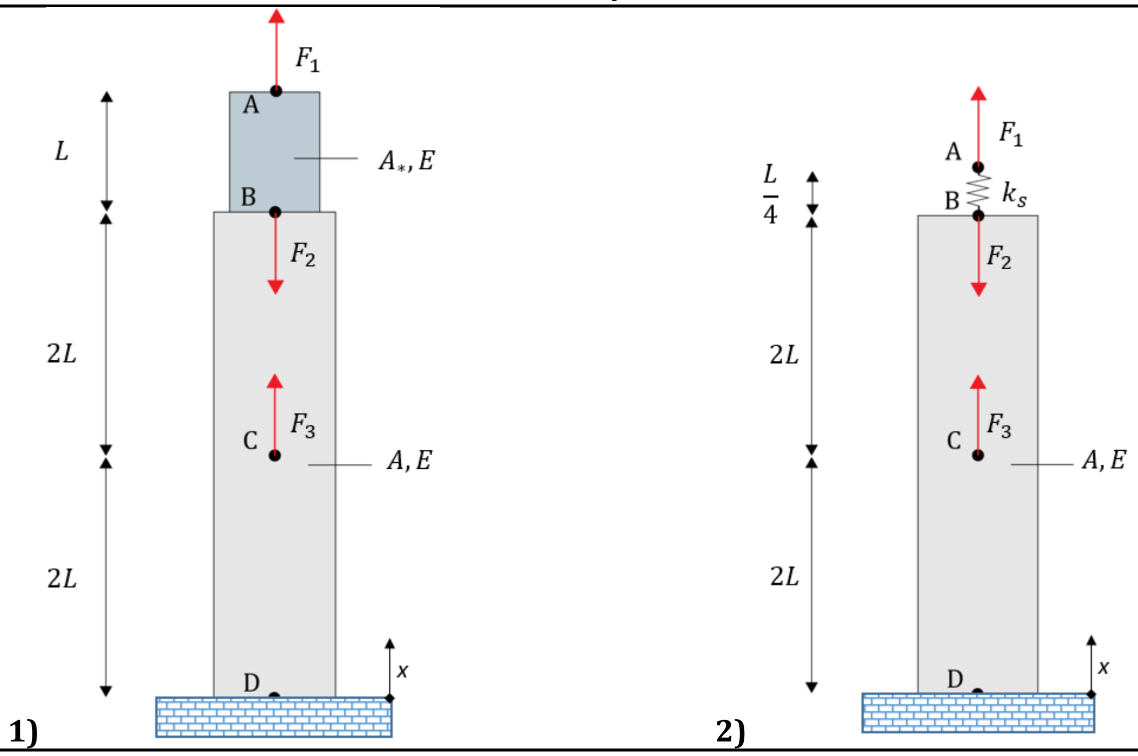
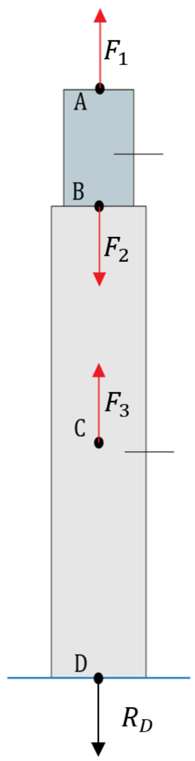


Figure 3b.4 | Composite posts: 1) bar/bar and 2) spring/bar

Solution 3b.4

a) The FBD is equal for both the systems since the spring can be seen as a bar



The structure can be divided as shown in figure 3b4.1 in order to calculate the internal forces:

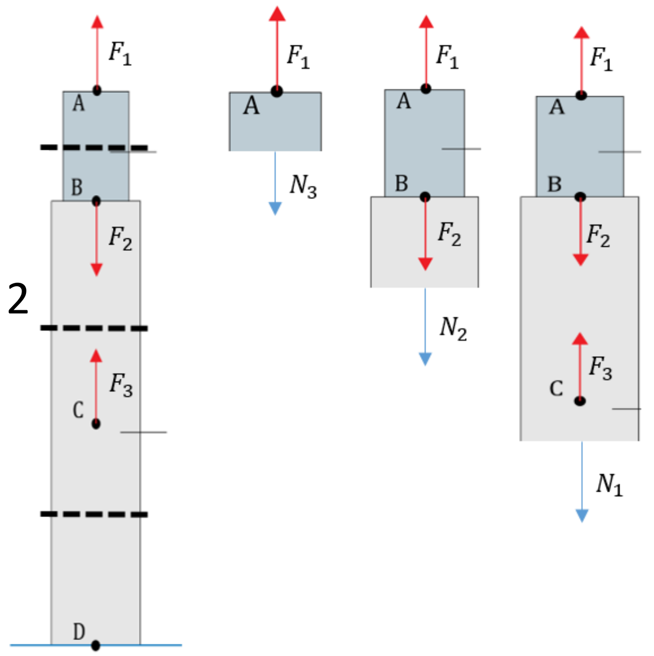


Figure 3b.4.1 | Composite posts: cuts to be done

<u>What are the Eqs. that are required?</u>

The stiffness of a segment CD:

$$k_{CD} = \frac{AE}{L_{CD}} \tag{1}$$

The stiffness of a segment CB:

$$k_{CB} = \frac{AE}{L_{CB}} \tag{2}$$

The stiffness of a segment AB, system a):

$$k_{AB} = \frac{A_* E}{L_{AB}} \tag{3}$$

While for system 2) is it equal to the stiffness of the spring k_s

Where A and A_* are the cross-section area of segments, L_{AB}, L_{CB} *and* L_{CD}, are the length and E the Young's modulus.

The internal force of a segment with respect to the displacement:

$$N = k\Delta \tag{4}$$

<u>Find Reaction at Point D</u>

$$\sum F_x = 0 \tag{5}$$

$$-R_D + 75kN - 45\,kN + 30\,kN = 0 \;\rightarrow R_D = 60\,kN \tag{6}$$

b) The deformation of the two systems can be calculated using

$$\delta_a = \sum_i \frac{N_i L_i}{A_i E} = \frac{1}{E}\left(\frac{N_1 * 2L}{A} + \frac{N_2 * 2L}{A} + \frac{N_3 * L}{A_*}\right) \tag{7}$$

$$\delta_b = \frac{1}{E}\left(\frac{N_1 * 2L}{A} + \frac{N_2 * 2L}{A}\right) + \frac{N_3}{k_s} \tag{8}$$

From figure 3b.4.1 can be seen that:

$$N_1 = 60\,kN$$
$$N_2 = -15\,kN$$
$$N_3 = 30\,kN$$

Plugging the numbers in (7) and (8) can be calculated the displacement:

$$\delta_a = \frac{1}{25 * 10^9}\left(\frac{60 * 10^3 * 0.2}{3 * 10^{-4}} - \frac{15 * 10^3 * 0.2}{3 * 10^{-4}} + \frac{30 * 10^3 * 0.1}{2 * 10^{-4}}\right) = 0.0018 = 1.8\,mm \tag{9}$$

$$\delta_b = \frac{1}{25 * 10^9}\left(\frac{60 * 10^3 * 0.2}{3 * 10^{-4}} - \frac{15 * 10^3 * 0.2}{3 * 10^{-4}}\right) + \frac{30 * 10^3}{10^8} = 0.0015 = 1.5\,mm \tag{10}$$

Exercise 3b.5 – Composed post

A post is composed of two different elements: a cube of height 3L between C and E (Young's modulus E_{CE}) and a square based tempered post with a of height 6L between A and C (Young's modulus E_{AC}). As shown in figure 3b.5, the section varies from A (side length 2L) to C (side length 3L) and two forces are applied to the system at point A and C. The amplitude of the force at point C is 2F and the amplitude of the force at point A is F. The materials are considered isotropic.

a) Draw the Free Body Diagram of the system and calculate the reaction force(s)
b) Calculate the value of the stress and strain of the post at section D
c) Calculate the value of the stress and strain of the post at section B
d) Calculate the deformation of the segment AC

$$\textit{Mathematical Hint}: \int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

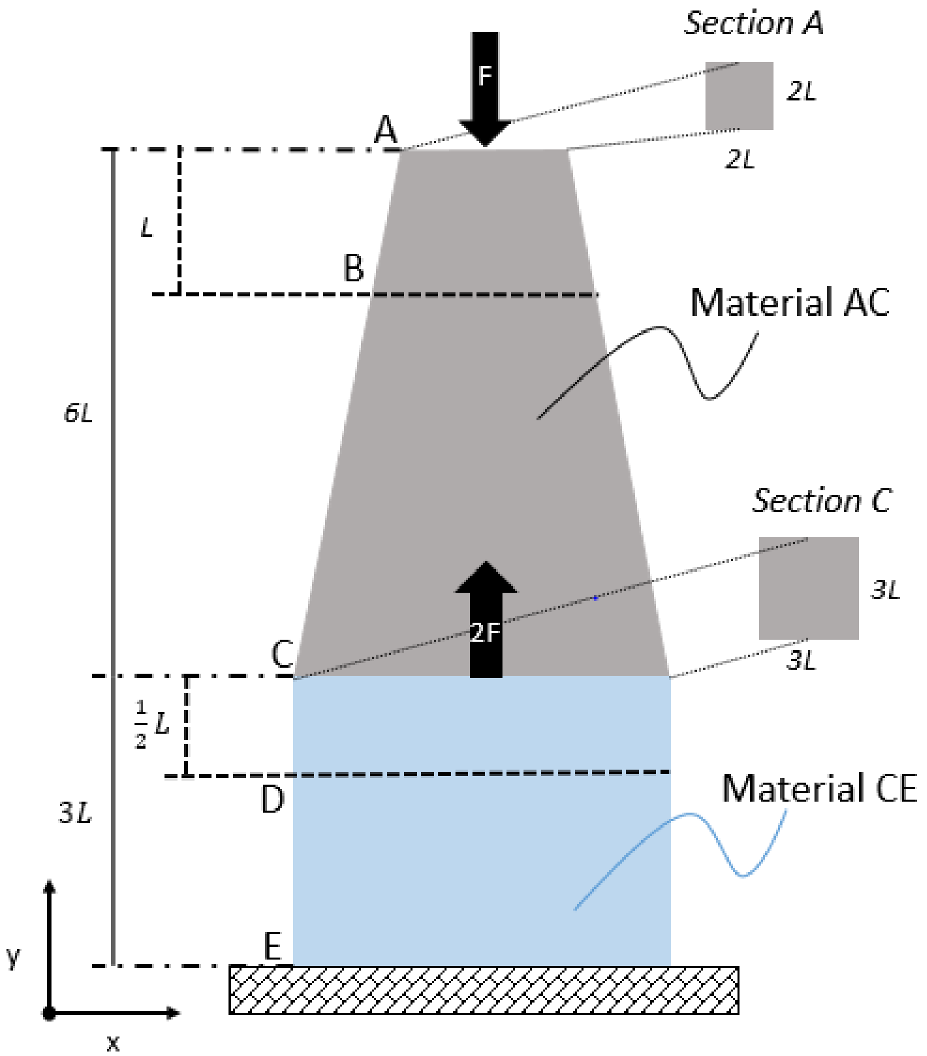
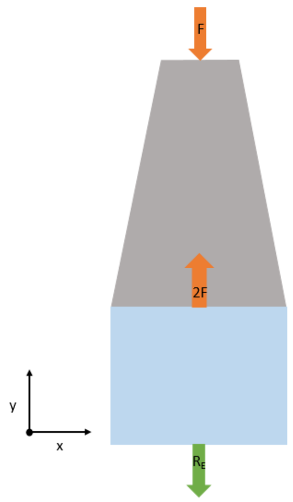


Figure 3b.5| Composed post

Solution 3b.5

a) Draw the Free Body Diagram of the system and calculate the reaction force(s).

Apply force equilibrium Eq. to the entire structure and evaluate R_E.



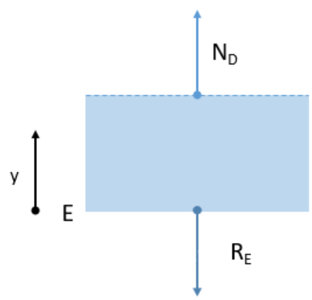
$$\sum F_y = 0 \rightarrow -R_E + 2F - F = 0 \rightarrow R_E = F \quad (1)$$

b) Calculate the value of the stress and strain of the post at section D

For the segment from E to D the area of the section is:

$$A = (3L)^2 = 9L^2$$

Segment DE



$$\sum F_y = 0 \rightarrow \quad -R_E + N_D = 0 \quad N_D = R_E = F \quad (2)$$

For the evaluation of the stress and the strain of the post at section D

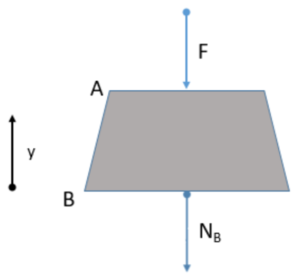
$$\sigma_D = \frac{N_D}{A_D} = \frac{F}{9L^2} \quad (3)$$

Strain is:

$$\varepsilon_D = \frac{\sigma_D}{E_{CE}} = \frac{F}{9L^2 E_{CE}} \quad (4)$$

c) Calculate the value of the stress and strain of the post at section B

Segment AB



$$\sum F_y = 0 \rightarrow -F + -N_B = 0 \rightarrow N_B = -F \tag{5}$$

For the segment from A to C the dimension of the square follows these formulas:

$$l(y) = L\left(2 + \frac{1}{6}\frac{y}{L}\right) \tag{6}$$

$$A(y) = L^2\left(2 + \frac{1}{6}\frac{y}{L}\right)^2 \tag{7}$$

or

$$l(y) = L\left(3 - \frac{1}{6}\frac{y}{L}\right)$$

$$A(y) = L^2\left(3 - \frac{1}{6}\frac{y}{L}\right)^2$$

The area of section B is:

$$A(L) = L^2\left(2 + \frac{L}{6L}\right)^2 = \frac{169}{36}L^2 \tag{8}$$

or

$$A(5L) = L^2\left(3 - \frac{5L}{6L}\right)^2 = \frac{169}{36}L^2$$

For the evaluation of the stress and the strain of the post at section B

$$\sigma_B = \frac{N_B}{A_B} = \frac{-F}{\frac{169}{36}L^2} = -\frac{36F}{169L^2} \tag{9}$$

Strain is:

$$\varepsilon_B = \frac{\sigma_B}{E_S} = -\frac{36F}{169L^2E_{AC}} \tag{10}$$

d) Calculate the deformation of the segment AC

Since the section varies along the axis is necessary to integrate between the tip of the post A and the section C.

The elongation can be evaluated with:

$$d\delta = \frac{-F dy}{E_{AC} A(y)} \tag{11}$$

By integrating (from A to C):

$$\delta_{AC} = \int_0^{6L} \frac{-F dy}{E_{AC} A(y)} \tag{12}$$

$$\delta_{AC} = \frac{-F}{E_{AC} L^2} \int_0^{6L} \frac{dy}{\left(2 + \frac{y}{6L}\right)^2} = -\frac{F}{E_{AC} L^2} \left[-\frac{6L}{2 + \frac{y}{6L}} \right]_0^{6L} \tag{13}$$

$$= -\frac{F}{E_{AC} L^2} (-2L + 3L) = -\frac{FL}{E_{AC} L^2} = -\frac{F}{E_{AC} L}$$

Or (from C to A)

$$\delta_{AC} = \int_0^{6L} \frac{-F dy}{E_{AC} A(y)}$$

$$\delta_{AC} = -\frac{F}{E_{AC} L^2} \int_0^{6L} \frac{dy}{\left(3 - \frac{y}{6L}\right)^2} = -\frac{F}{E_{AC} L^2} \left[-\frac{6L}{3 - \frac{y}{6L}} \right]_0^{6L} = -\frac{F}{E_{AC} L^2} (-2L + 3L) = -\frac{FL}{E_{AC} L^2} = -\frac{F}{E_{AC} L}$$

The segment AC is compressed and shortens of a quantity equal to: $\frac{F}{E_{AC} L}$

